THE FUNDAMENTAL THEOREM OF CARDINAL ARITHMETIC

Theorem 1. *$\kappa^2 = \kappa$ for every infinite cardinal κ.*

We need to show that there is a bijection between κ and $\kappa \times \kappa$. Let us give two proofs. The first one relies on the fact that for every well-ordering $(L, <)$ there is a unique ordinal α such that $(L, <) \cong (\alpha, \in)$. The second proof is even more elementary, it is based on Zorn's lemma. Both of the proofs uses transfinite induction on κ. To start the induction let $F(n) = (\ell, m)$ for the unique (ℓ, m) with $n = 2^\ell \cdot (2m+1) - 1$. This F is a bijection between ω and $\omega \times \omega$. Let κ be an uncountable cardinal.

First proof: We define a well-order on $\kappa \times \kappa$ by letting $(\alpha_0, \beta_0) \prec (\alpha_1, \beta_1)$ iff

$$\begin{aligned}
&\max\{\alpha_0, \beta_0\} < \max\{\alpha_1, \beta_1\} \text{ or}\\
&\max\{\alpha_0, \beta_0\} = \max\{\alpha_1, \beta_1\} \text{ and } \alpha_0 < \alpha_1 \text{ or}\\
&\max\{\alpha_0, \beta_0\} = \max\{\alpha_1, \beta_1\} \text{ and } \alpha_0 = \alpha_1 \text{ and } \beta_0 < \beta_1.
\end{aligned}$$

(Check that this is indeed a well-order!) Then there is an ordinal λ and an order-preserving bijection f from $(\lambda, \in)$ to $(\kappa \times \kappa, \prec)$. We claim that $\lambda = \kappa$. Since $|\kappa \times \kappa| \geq \kappa$ obviously holds, we know that λ must be at least κ. Suppose for a contradiction that $\lambda > \kappa$ and let $(\alpha, \beta) := f(\kappa)$. Then on the one hand, there are κ many elements below (α, β) with respect to $\prec$ (namely $f(\gamma)$ for $\gamma < \kappa\}$). On the other hand, every such element has only coordinates smaller than $\max\{\alpha, \beta\} < \kappa$ so there are at most $|\max\{\alpha, \beta\}|^2$ such elements. By induction $|\max\{\alpha, \beta\}|^2 = |\max\{\alpha, \beta\}| < \kappa$, a contradiction. $\square$

Second proof: Consider the set

$$P := \{f : \ f \text{ is a function with } \omega \subseteq D_f \subseteq \kappa, \ R_f = D_f \times D_f\}$$

partially ordered by $\subseteq$ (i.e. g is larger than f iff $D_g \supseteq D_f$ and $f(\alpha) = g(\alpha)$ for $\alpha \in D_f$). The conditions of Zorn's lemma hold: for a non-empty chain the union of its elements is an upper bound in $(P, \subseteq)$, for the empty chain any element is an upper bound ($P \neq \emptyset$ because $F \in P$). Let h be a maximal element in $(P, \subseteq)$. If $|D_h| = \kappa$, then by using a bijection between D_h and $|D_h|$ we can obtain a desired $\kappa \to \kappa \times \kappa$ bijection from h. Suppose for a contradiction that $|D_h| < \kappa$. Then we must have $|\kappa \setminus D_h| = \kappa$. Indeed, if we had $|\kappa \setminus D_h| < \kappa$, then

$$\kappa = |D_h| + |\kappa \setminus D_h| \leq 2 \cdot \max\{|D_h|, |\kappa \setminus D_h|\} \leq \max\{|D_h|, |\kappa \setminus D_h|\}^2 \overset{\text{induc.}}{=} \max\{|D_h|, |\kappa \setminus D_h|\} < \kappa$$

which is impossible, so $|\kappa \setminus D_h| = \kappa$, as we claimed.

Let $X \subseteq \kappa \setminus D_h$ with $|D_h| = |X| =: \lambda$. We show that there is a bijection h' between X and $X^2 \cup (D_h \times X) \cup (X \cup D_h)$. Note that if it is done, then $h \cup h' : D_h \cup X \to (D_h \cup X) \times (D_h \cup X)$ is a bijection contradicting the maximality of h. To ensure the existence of h', we prove by using the induction hypothesis for λ, that $|X^2 \cup (D_h \times X) \cup (X \cup D_h)| = \lambda$:

$$\left|X^2 \cup (D_h \times X) \cup (X \times D_h)\right| =$$
$$\left|X^2\right| + |D_h \times X| + |X \times D_h| =$$
$$|X|^2 + |D_h| \cdot |X| + |X| \cdot |D_h| =$$
$$\lambda^2 + \lambda^2 + \lambda^2 = 3 \cdot \lambda^2 = 3 \cdot \lambda \leq \lambda \cdot \lambda = \lambda$$

□